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ON HESSIANS AND STEINERIANS OF HIGHER ORDERS IN GEOMETRY OF ONE DIMENSION.

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In a paper entitled, *On Hessians, Jacobians, Steinerians, etc., in Geometry of one Dimension*, published in Vol. III, No. 2 of the *Kansas University Quarterly*, the writer discussed the properties of the Hessian and Steinerian of a one-dimensional geometric form. The substance of that discussion was summed up in the following theorem (Theorem IV, page 106).

If the first polar of a point S with respect to a non-singular quantic U has a double point at H, then the polar quadratic of H has a double point at S; and vice versa. *The totality of all points H is the Hessian of U, and the totality of all points S is the Steinerian of U.*

It will be observed that the above-mentioned discussion was limited to the consideration of double points on first and $(n-2)$th polars of U. The object of the present paper is to enlarge the range of that discussion and to extend it to the consideration of double points of the intermediate polars of U, and thus to introduce the conception of Hessians and Steinerians of higher orders. The following definitions will serve as a starting point for this discussion:

Definitions:—The rth *Hessian of a non-singular quantic U is the totality of double points on all* rth *polars of U.*

The rth *Steinerian of a non-singular quantic U is the totality of points whose* rth *polars with respect to U have double points.*

According to these definitions the Hessian and Steinerian of the paper referred to above are respectively the first Hessian and the first Steinerian of U. We shall first show how to obtain expressions for this series of Hessians.

As before let $U=0$ be a non-singular quantic of the nth degree, homogeneous in x and y. The first polar of a point x_1, y_1 with respect to U is given by the operation

$$(P)U=\left[x_1\frac{d}{dx}+y_1\frac{d}{dy}\right]U=0\,.$$

The totality of double points on all first polars is found by eliminating x_1 and

y_1 from the x and y derivatives of $(P)\,U$. This resultant is the well known form

$$(H)\,U = \begin{vmatrix} \dfrac{d^2U}{dx^2} & \dfrac{d^2U}{dxdy} \\[2ex] \dfrac{d^2U}{dxdy} & \dfrac{d^2U}{dy^2} \end{vmatrix} = 0\,.$$

This is the equation of the first Hessian of U.

The equation of the second Hessian of U is obtained by eliminating x_1 and y_1 from the x and y derivatives of the second polars of U, viz:

$$(P)^2\,U = x_1^{\,2}\frac{d^2U}{dx^2} + 2x_1y_1\frac{d^2U}{dxdy} + y_1^{\,2}\frac{d^2U}{dy^2} = 0\,.$$

This resultant is given by Sylvester's method in the form of a determinant as follows:

$$(H_2)\,U = \begin{vmatrix} \dfrac{d^3U}{dx^3} & 2\dfrac{d^3U}{dx^2dy} & \dfrac{d^3U}{dxdy^2} & 0 \\[2ex] 0 & \dfrac{d^3U}{dx^3} & 2\dfrac{d^3U}{dx^2dy} & \dfrac{d^3U}{dxdy^2} \\[2ex] \dfrac{d^3U}{dx^2dy} & 2\dfrac{d^3U}{dxdy^2} & \dfrac{d^3U}{dy^3} & 0 \\[2ex] 0 & \dfrac{d^3U}{dx^2dy} & 2\dfrac{d^3U}{dxdy^2} & \dfrac{d^3U}{dy^3} \end{vmatrix} = 0\,.$$

In like manner the third, fourth, . . . , and rth Hessian of U may be obtained. The rth Hessian of U is obtained by eliminating x_1 and y_1 from the x and y derivatives of $(P)^r\,U = 0$. This resultant is given by Sylvester's method as a determinant of the order $2r$. The constitutents of this determinant are $(r + 1)$th derivatives of U. Since the $(r + 1)$th derivatives of U are of order $(n - r - 1)$, it follows that the rth Hessian of U is of order $2r\,(n - r - 1)$ in x and y.

Since U is of order n, the last polar of U is the $(n - r)$th and is of the first order in x and y. It is a single point. The $(n - 2)$th polar is a quadratic and is the last one that can have a double point. Hence the last Hessian of U is the $(n - 2)$th; this is the maximum value of r.

Theorem:—A non-singular quantic U of order n has a series of $(n - 2)$ *Hessians. The order of the* rth *Hessian is* $2r\,(n - r - 1)$.

From the above formula we see that the order of the first Hessian is

$2(n-2)$; the order of the second is $4(n-3)$, etc., the order of the last or $(n-2)$th Hessian $2(n-2)$, the same as the first. The order of the $(n-3)$th Hessian is $4(n-3)$, the same as the second. This law connecting the orders of the Hessians equally removed from the extremes of the series is easily seen to hold true in general. The order of the rth Hessian is equal to that of the $(n-r-1)$th; i. e. the order of the rth Hessian is not changed when r is changed into $(n-r-1)$, a self-evident fact.

Our series of Hessians of U will be designated by $H_1, H_2, H_3, \ldots, H_{n-3}, H_{n-2}$.

We turn now to the consideration of the Steinerians of U. The first polars of U are given by $x_1\frac{dU}{dx}+y_1\frac{dU}{dy}=0$, where (x_1, y_1) is any point on the line. If this first polar have a double point, its discriminant vanishes. When this first polar is arranged according to powers of x and y, the coefficients all contain both x_1 and y_1 in the first degree. Hence equating to zero the discriminant of $(P)U=0$ we obtain the equation of the first Steinerian of U, which is of degree $2(n-2)$ in x_1 and y_1.

In like manner the equation of the second Steinerian of U is obtained by equating to zero the discriminant of $(P)^2U=0$. This gives an equation in x_1 and y_1 of degree $4(n-3)$. In general the rth Steinerian of U is gotten by equating to zero the discriminant of the rth polar of U, $(P)^rU=0$; it is of degree $2r(n-r-1)$ in x_1 and y_1.

The $(n-1)$th polar of U is of the first degree in x and y and consequently has no discriminant. The $(n-2)$th polar is a quadratic in x and y and is the last polar that can have a discriminant. Consequently the series of Steinerian of U ends with the $(n-2)$th.

Let us now examine into the algebraic form of these Steinerians of U, beginning with the $(n-2)$th. The $(n-2)$th polar of (x_1, y_1) with respect to U when arranged according to powers of x and y may be written

$$(P)^2U_1 = x^2\frac{d^2U_1}{dx_1^2}+2xy\frac{d^2U_1}{dx_1dy_1}+y^2\frac{d^2U_1}{dy_1^2}=0\,.$$

The discriminant of this expression equated to zero gives the equation of the $(n-2)$th Steinerian; thus

$$(S_{n-2})\,U = \begin{vmatrix} \dfrac{d^2U_1}{dx_1^2} & \dfrac{d^2U_1}{dx_1dy_1} \\[2ex] \dfrac{d^2U_1}{dx_1dy_1} & \dfrac{d^2U_1}{dy_1^2} \end{vmatrix} = 0\,.$$

In this equation x_1 and y_1 are the variables; thus we see that the $(n - 2)$th Steinerian of U is identical with the first Hessian of U. This fact is implied in the theorem quoted at the beginning of this article.

The $(n - 3)$th Steinerian is obtained by equating to zero the discriminant of

$$(P)^3 U = x^3 \frac{d^3 U_1}{dx_1^3} + 3x^2 y \frac{d^3 U_1}{dx_1^2 dy} + 3xy^2 \frac{d^3 U_1}{dx_1 dy^2} + y^3 \frac{d^3 U_1}{dy_1^3} = 0 .$$

This written in determinant form is as follows:

$$(S_{n-3})\, U = \begin{vmatrix} \frac{d^3 U_1}{dx_1^3} & 2\frac{d^3 U_1}{dx_1^2 dy_1} & \frac{d^3 U_1}{dx_1 dy_1^2} & 0 \\ 0 & \frac{d^3 U_1}{dx_1^3} & 2\frac{d^3 U}{dx_1^2 dy_1} & \frac{d^3 U_1}{dx_1 dy_1^2} \\ \frac{d^3 U_1}{dx_1^2 dy_1} & 2\frac{d^3 U_1}{dx_1 dy_1^2} & \frac{d^3 U_1}{dy_1^3} & 0 \\ 0 & \frac{d^3 U_1}{dx_1^2 dy_1} & 2\frac{d^3 U_1}{dx_1 dy_1^2} & \frac{d^3 U_1}{dy_1^3} \end{vmatrix} = 0 .$$

We see at once that the $(n - 3)$th Steinerian of U is identical with the second Hessian of U.

In like manner it may be shown in general that the $(n - r - 1)$th Steinerian of U is identical with the rth Hessian of U. Writing the identical quantities under each other the two series and their relation may be shown as follows:

$$\begin{array}{cccccc} H_1, & H_2, & H_3, & \ldots, & H_{n-3}, & H_{n-2} . \\ S_{n-2}, & S_{n-3}, & S_{n-4}, & & S_2, & S_1 . \end{array}$$

The fact of the identity of these series of Hessians and Steinerians enables us to state the following theorem, which is a generalization of that quoted at the beginning of this paper:

Theorem:—If the rth *polar of a point S has a double point at H, then the* $(n - r - 1)$th *polar of H has a double point at S; and* vice versa. *The totality of all points H is the* rth *Hessian of U, and the totality of all points S is the* $(n - r - 1)$th *Steinerian of U.*

Since the rth Hessian is identical with the $(n - r - 1)$th Steinerian, and the $(n - r - 1)$th Hessian is identical with the rth Steinerian, we see that the rth and the $(n - r - 1)$th Hessians are so related to each other that there exists between them a one-to-one correspondence. To each point of one there corresponds a definite point of the other. When $r = n - r - 1$, i. e. when $r = (n - 1)/2$, the points of the Hessian given by this value of r correspond to each other in pairs. Since r and n are integers, this relation can hold only when n is odd. In such a case the middle Hessian of the series is both rth Hessian and rth Steinerian. *Therefore when n is odd, the $(n - 1)/2$th polar of a point H_1 on the $(n - 1)/2$th Hessian of U has a double point at H_2 on the same Hessian; and the $(n - 1)/2$th polar of H_2 has a double point at H_1.*

This theorem finds an interesting application in the theory of canonical forms of binary quantics of odd degree. Take for example the cubic

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0.$$

This quantic is of order three, and since $n - 2 = 1$, the cubic has but one Hessian and but one Steinerian; these must be identical. This Hesso-Steinerian is of degree $2(n - 2)$ or 2, and consists of two points A and B. The first polar of A has a double point at B, and the first polar of B has a double point at A. We may take these two points as the ground points of the system of coordinates. The coordinates of A are $(x_1, 0)$ and of B are $(0, y_1)$. The first polar of A is $ax^2 + 2bxy + cy^2 = 0$; if this has a double point at $x = 0$, b and c must vanish. In like manner the first polar of B is $bx^2 + 2cxy + dy^2 = 0$; if this has a double point at $y = 0$, b and c must vanish. Hence we see that if the points of the Hessian be taken for the ground points, the two middle terms of the cubic must vanish; and the cubic reduces to the form $ax^3 + dy^3 = 0$.

The same principle may be applied to the quintic. The second Hessian is identical with the second Steinerian. Let a pair of corresponding points on this second Hessian be chosen as the ground points. The second polar of the ground point A, $(x_1, 0)$, will then have a double point at B; and the second polar of B, $(0, y_1)$, will then have a double point at A. Let the quintic be written

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5 = 0.$$

The second polar of $(x_1, 0)$ is $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$; in order that this should have $x = 0$ for a double point, it is necessary that c and d should vanish. The second polar of $(0, y_1)$ is $cx^3 + 3dx^2y + 3exy^2 + fy^3 = 0$; this will have a double point at $y = 0$, when c and d both vanish. Therefore the quintic can always be brought to the form

$$ax^5 + 5bx^4y + 5exy^4 + dy^5 = 0,$$

by choosing for ground points a pair of corresponding points on the second Hessian. This second Hessian of the quintic is of degree $4(n - 3)$ or 8;

hence there are four distinct pairs of corresponding points on this second Hessian. This choosing of a new pair of ground points is equivalent to a linear transformation of the quintic. Hence

Every non-singular quintic can be brought by a linear transformation to the canonical form, $ax^5 + 5bx^4y + 5exy^4 + fy^5 = 0$, *in four distinct ways.*

In like manner it may be shown in general that every non-singular quantic of odd order n can be linearly transformed so that the two middle terms shall vanish. Since the middle Hessian of the series is the $(n-1)/2$th, and since the order of the rth Hessian is $2r(n-r-1)$, it follows that the $(n-1)/2$th Hessian is of order $2\left[\frac{n-1}{2}\right]\left[n-\frac{n-1}{2}-1\right]$ or $\frac{(n-1)^2}{2}$; i. e. it consists of $(n-1)^2/2$ points or of $\left[\frac{n-1}{2}\right]^2$ pairs of points. Hence we may formulate the general theorem

Every non-singular quantic of odd degree n may be linearly transformed so that its two middle terms shall vanish; this may be done in $\left[\frac{n-1}{2}\right]^2$ *distinct ways.*

The above series of $(n-2)$ Hessians of U are covariants of U; we have already found expressions for these in determinant form in terms of the x and y derivatives of U. It remains to find expressions for these covariants, if possible, in terms of simpler covariants of U.

The first Hessian of U is one of the fundamental covariants of U and not expressible in terms of simpler ones. Since the equation of the first Hessian of U is obtained in the form of the discriminant of a quadratic whose coefficients are second derivatives of U, it follows that the coefficient of the first term, sometimes called its source, is always $(ac-b^2)$. If we designate the discriminant of the quadratic by H, then the first Hessian of the $n$ic may be designated by $H(xy)$.

The Jacobian of U and its first Hessian is another fundamental covariant of U, and therefore not expressible in terms of simpler ones. The coefficient of its first term is always $(a^2d - 3abc + 2b^3)$. Designating this by G, the Jacobian of U may be written $G(xy)$.

The equation of the second Hessian of U is obtained in the form of the discriminant of a cubic whose coefficients are third derivatives of U. Its source, i. e. the coefficient of its first term, is therefore always of the form $\varDelta = \frac{G^2 + 4H^3}{a^2}$. By a well known principle in the Theory of Algebraic Forms any relation such as the last, which holds between the sources of several covariants, holds also between the covariants themselves. Let U be an $n$ic, its source is a; that of its first Hessian is H; that of its Jacobian is G; and that of its second Hessian is $\varDelta$. Therefore we have the relation

$$G^2(xy) + 4H^3(xy) = U^2\varDelta(xy).$$

The square of the Jacobian of any quantic plus four times the cube of its first Hessian is always divisible by the square of the quantic itself; and the quotient is the second Hessian of the quantic.

The second Hessian of U_1 $\varDelta(xy)$, is not a fundamental covariant of U and can be expressed in terms of still simpler ones. The lowest quantic which has a second Hessian is the quartic. The two fundamental invariants of the quartic are $I = ae - 4bd + 3c^2$ and $J = ace + 2bcd - ad^2 - eb^2 - c^3$. The following relation holds between the invariants and the semi-invariants of the quartic:

$$G^2 + 4H^3 = a^2(HI - aJ).$$

Considering these quantities as sources of covariants of any quantic, we have the relation:

$$G^2(xy) + 4H^3(xy) = U^2[H_1(xy)I(xy) - UJ(xy)].$$

Comparing this result with the one above we see at once that the second Hessian

$$H_2(xy) = \varDelta(xy) = H_1(xy)I(xy) - UJ(xy).$$

In the case of the quartic $I(xy)$ and $J(xy)$ become the invariants I and J, and hence the second Hessian of the quartic is as follows:

$$H_2 = H_1I - UJ.$$

The expression for the third Hessian of U in terms of simpler covariants is easily obtained from the form of the discriminant of the quartic. The discriminant of the quartic is $I^3 - 27J^2$. These invariants of the quartic I and J are sources of irreducible covariants for all forms of U higher than the quartic. Therefore the third Hessian of U is always expressible in form

$$I^3(xy) - 27J^2(xy) = 0.$$

The quintic is the lowest form which has a third Hessian.

We are now in position to generalize the method employed above for the expression of the higher Hessians of U in terms of irreducible covariants of U. The lowest $n$ic which has an rth Hessian is the quantic U_n for which $n = r + 2$. The rth Hessian of U_n is in the form of the discriminant of the quantic of order $(r + 1)$. If we know the discriminant of the $(r + 1)$ic in terms of its fundamental invariants and semi-invariants, then we can write down at once the rth Hessian of the $(r+2)$ic and of all higher forms.

We give in conclusion a table showing for the lower binary quantics expressions for the series of Hessians in terms of the fundamental covariants of the forms. The first line of the table gives the sources of the successive Hessians of U.

	U	1st H.	2d H.	3d H.	4th H.	5th H.	6th H.
	a	$(ac - b^2)$	$\Delta = \frac{G^2 + 4H^3}{a^2}$	$I^3 - 27J^2$	$J_4^2 - 128J_8$		
$n = 1$	a						
$n = 2$	a	Disc.					
$n = 3$	a	H_1	Disc. $H_1I - UJ$.				
$n = 4$	a	H_1	H_2	Disc.			
$n = 5$	a	H_1	H_2	H_3	Disc.		
$n = 6$	a	H_1	H_2	H_3	H_4	Disc.	
.	.	.	.	.	.	H_5	Disc.
.	.	.	.	.	.	.	H_6
.	.	.	.	.	.	.	.

FEB. 29, 1896.